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Integrable quantum chain and the representation of the quantum group $SU_q(2)$

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Abstract. We discuss the *XXZ* spin chain with certain boundary terms and the representation of quantum group $SU_q(2)$. It is shown that Bethe ansatz states are highest-weight states of $SU_q(2)$. With a generic q we construct the irreducible representations of the quantum group. For q being a root of unity, we show that there are new Bethe ansatz states, which coincide with null states of $SU_q(2)$. By taking certain limits we can derive the state $|b\rangle$, which is necessary for constructing the indecomposable but reducible representations of $SU_q(2)$ and for the completeness of the state space. In this case the Hamiltonian may not be completely diagonalized.

1. Introduction

Recently, one-dimensional exactly solvable statistical models have been extensively studied, and Heisenberg spin chain systems are of particular interest [1–6]. It has been shown that the isotropic *XXX* model corresponds to the rational solutions of the Yang–Baxter equation [4, 5]. The model possesses $SU_q(2)$ symmetry and is exactly solvable [6]. Yang and Yang [7] studied its ground and excited states. Izergin and Korepin [8] have shown that for Bethe ansatz (BA) states associated with the *XXX* model, the impulsions are distinct. It has been shown that the *XXZ* model with periodic conditions and the six-vertex model are equivalent [3–5]. For the *XXZ* model with particular boundary conditions, Alcaraz *et al* [9] and Sklyanin [10] have obtained the energy and eigenstates, using the BA and QISM respectively. It has been shown that the model has $SU_q(2)$ symmetry and may be related to the representation of $SU_q(2)$.

For the *XXZ* model with particular conditions, the Hamiltonian can be expressed as a linear combination of the elements of Temperley–Lieb algebra [1]. The impulsion ks in BA states are generically all distinct and satisfy the BA equation. In this paper we show that BA states are highest-weight states (HWS) of $SU_q(2)$. We construct the representation of $SU_q(2)$ and discuss its completeness. We then look for BA states with some of its impulsion ks identical. For the parameter $SU_q(2)$, q being a root of unity, we find that BA states may have identical ks with $e^{ik}=q$.

In this case some states in the $SU_q(2)$ representation will degenerate into null states, and coincide with such BA states [11]. Thus the representation space is no longer complete due to the coincidence of two originally independent states. We present a new approach to overcome this difficulty and obtain an indecomposable (type I) representation of $SU_q(2)$.

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2. *XXZ* model and $SU_q(2)$

Quantum $SU_q(2)$ algebra is a generalization of classical SU(2) algebra, with generators satisfying

$$[S_z, S_\pm] = \pm S_\pm \qquad [S_+, S_-] = [2S_z] \tag{2.1}$$

where $[X] \equiv (q^x - q^{-x})/(q - q^{-1})$. The above expressions degenerate into classical SU(2) algebra in the limit $q \to 1$. We can define the co-product, co-unity and antipode for the algebra given by (2.1) as

$$\begin{aligned} &\Delta(q^{\pm S_z}) = q^{\pm S_z} \otimes q^{\pm S_z} && \Delta(S_\pm) = q^{S_z} \otimes S_\pm + S_\pm \otimes q^{-S_z} \\ &\gamma(q^{\pm S_z}) = q^{\mp S_z} && \gamma(S_\pm) = -q^{\pm 1} S_\pm \\ &\varepsilon(q^{\pm S_z}) = 1 && \varepsilon(S_\pm) = 0 \end{aligned} \tag{2.2}$$

and we have the direct product of N fundamental representations of $SU_q(2)$:

$$\begin{aligned} S_z &= \sum_{i=1}^{N} \sigma_i^z/2 \\ S_\pm &= \sum_{i=1}^{N} q^{\sigma_1^z/2} \otimes \ldots \otimes q^{\sigma_{i-1}^z/2} \otimes \sigma_i^\pm/2 \otimes q^{-\sigma_{i+1}^z/2} \otimes \ldots \otimes q^{-\sigma_N^z/2} \end{aligned} \tag{2.3}$$

where σ_is are Pauli matrices, and the index i denotes the space. The centre of the algebra is

$$S^2 = S_- S_+ + [S_z + \tfrac{1}{2}]^2 - [\tfrac{1}{2}]^2 \tag{2.4}$$

and, further, we have the following relation:

$$[S_+^n, S_-^m] = S_-^{m-n} \frac{[m]!}{[m-n]!} \prod_{k=1}^{n} [2S_z - m + k] \tag{2.5}$$

with both sides acting on $\ker(S_+)$, where

$$[m]! \equiv [m][m-1] \ldots [2][1] \qquad [0]! = 1.$$

The Hamiltonian for the *XXZ* model with particular boundary conditions is given by

$$H = \sum_{i=1}^{N-1} \left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \frac{q+q^{-1}}{2} \sigma_i^z \sigma_{i+1}^z \right) + \frac{q-q^{-1}}{2} (\sigma_1^z - \sigma_N^z) \tag{2.6}$$

or

$$H = \sum_{i=1}^{N-1} [(q+q^{-1})/2 - 2e_i]$$

with

$$e_i = \frac{-1}{2} \left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \frac{q+q^{-1}}{2} \sigma_i^z \sigma_{i+1}^z + \frac{q-q^{-1}}{2} (\sigma_i^z - \sigma_{i+1}^z) \right) + \frac{q+q^{-1}}{4} \tag{2.7}$$

where e_i is the generator of Temperley–Lieb algebra [1, 10]. Since Temperley–Lieb algebra commutes with $SU_q(2)$, it is easy to show that

$$[H, S_z]=0 \qquad [H, S_\pm]=0 \tag{2.8}$$

i.e. the spin chain described by the Hamiltonian H possesses quantum $SU_q(2)$ symmetry. From (2.3) we can thus represent a state by eigenvalues of H and S_z. Alcaraz has obtained the eigenstates and their corresponding eigenvalues of this system [9]. For a configuration with n spins down ($N-n$ up), the eigenstate of H is

$$|n\rangle_{BA}=\sum_{\{x_i\}} f(x_1,\ldots,x_n)|x,\ldots,x_n\rangle \tag{2.9}$$

where x denotes the position of the downward spin $1\leqslant x_1<\ldots<x\leqslant N$, ($n\leqslant N/2$), and

$$f=\sum_P \varepsilon_p A(k_{p_1},\ldots,k_{p_n})\exp[\mathrm{i}(k_{p_1}x_1+\ldots+k_{p_n}x_n)] \tag{2.10}$$

the summation above is over all permutations and negations of impulsions k; ε_p changes sign at each transformation. The coefficients A are

$$\begin{aligned} A(k_1,\ldots,k_n)&=\prod_{j=1}^{n}\beta(-k_j)\prod_{1\leqslant j<l\leqslant n}B(-k_j,k_l)\exp(-\mathrm{i}k_l)\\ \beta(k)&=(1-q^{-\mathrm{i}k})\,\mathrm{e}^{\mathrm{i}(N+1)k}\\ B(k_1,k_2)&=[1-(q+q^{-1})\,\mathrm{e}^{\mathrm{i}k_2}+\mathrm{e}^{\mathrm{i}(k_1+k_2)}][1-(q+q^{-1})\,\mathrm{e}^{-\mathrm{i}k_1}\,\mathrm{e}^{\mathrm{i}(k_2-k_1)}]\\ &=(\mathrm{e}^{\mathrm{i}k_1}+\mathrm{e}^{-\mathrm{i}k_2}-q-q^{-1})(\mathrm{e}^{\mathrm{i}k_1}+\mathrm{e}^{\mathrm{i}k_2}-q-q^{-1})\,\mathrm{e}^{\mathrm{i}(k_2-k_1)} \end{aligned} \tag{2.11}$$

and the impulsion ks satisfy the BA equations

$$\frac{\alpha(k_j)\beta(k_j)}{\alpha(-k_j)\beta(-k_j)}=\prod_{\substack{l=1\\ l\neq j}}^{n}\frac{B(-k_j,k_l)}{B(k_j,k_l)} \qquad (j=1,2,\ldots,n) \tag{2.12}$$

with $\alpha(k)=1-q^{-1}\exp(-\mathrm{i}k)$.

The corresponding energy eigenvalue is

$$E_n=(N-1)(q+q^{-1})/2+4\sum_{j=1}^{n}[\cos k_j-(q+q^{-1})/2]. \tag{2.13}$$

3. The irreducible representations of $SU_q(2)$

In the preceding section, the $SU_q(2)$ symmetry of the system is presented. The representation space of $SU_q(2)$ can be obtained by applying S_+ and S_- on all the BA states. We show that BA states are HWSs of $SU_q(2)$, as pointed out by Pasquier and Saleur [1], and thus the whole state space is obtained by acting on BA states with S_-.

Take a BA state with $n+1$ spin down as

$$|n+1\rangle_{BA}=\sum_{\{x_i\}} f(x_1,\ldots,x_{n+1})|x_1,\ldots,x_{n+1}\rangle. \tag{3.1}$$

When S_+ is applied it flips a downward spin upward, thus

$$S_+|n+1\rangle_{BA}\equiv|\beta\rangle=\sum_{\{x_i\}} g(x_1,\ldots,x_n)|x_1,\ldots,x_n\rangle. \tag{3.2}$$

From the definition of S_+ we have, after a straightforward calculation,

$$
\begin{aligned}
g(x_1,\ldots,x_n)|x_1,\ldots,x_n\rangle
&= \sum_{x=x_n+1}^{N} f(x_1,\ldots,x_n,x)q^{x-(N+2n+1)/2}|x_1,\ldots,x_n\rangle \\
&\quad+\sum_{l=2}^{n-1}\sum_{x=x_{l-1}+1}^{x_l-1} f(x_1,\ldots,x_{l-1},x,x_l,\ldots,x_n)q^{x-2l-(N-2n-3)/2}|x_1,\ldots,x_n\rangle \\
&\quad+\sum_{x=1}^{x_1-1} f(x,x_1,\ldots,x_n)q^{x-(N-2n+1)/2}|x_1,\ldots,x_n\rangle.
\end{aligned}
\tag{3.3}
$$

Consequently we have

$$
\begin{aligned}
q^{(N-2n+1)/2}g(x_1,\ldots,x_n)
&= \sum_{x=x_n+1}^{N} f(x_1,\ldots,x_n,x)q^{x-2n}+\sum_{x=1}^{x_1-1} f(x,x_1,\ldots,x_n)q^{x} \\
&\quad+\sum_{l=2}^{n-1}\sum_{x=x_{l-1}+1}^{x_l-1} f(x_1,\ldots,x_{l-1},x,x_l,\ldots,x_n)q^{x-2(l-1)}.
\end{aligned}
\tag{3.4}
$$

Taking into account (2.10) and the relation

$$
\sum_{x=x_1}^{x_2} q^x\, \mathrm{e}^{\mathrm{i}kx}=\frac{1}{1-q\,\mathrm{e}^{\mathrm{i}k}}\left(q^{x_1}\,\mathrm{e}^{\mathrm{i}kx_1}-q^{x_2+1}\,\mathrm{e}^{\mathrm{i}k(x_2+1)}\right)
\tag{3.5}
$$

(3.4) is written as

$$
\begin{aligned}
q^{(N-2n+1)/2}g(x_1,\ldots,x_n)
&=\sum_p \varepsilon_p A(k_{p_1},\ldots,k_{p_{n+1}})q^{-2n}\exp\left(\mathrm{i}\sum_{j=1}^{n} k_{p_j}x_j\right) \\
&\quad\times\frac{[q\exp(\mathrm{i}k_{p_{n+1}})]^{x_n+1}-[q\exp(\mathrm{i}k_{p_{n+1}})]^{N+1}}{1-q\exp(\mathrm{i}k_{p_{n+1}})} \\
&\quad+\sum_{l=2}^{n-1}\sum_p \varepsilon_p A(k_{p_1},\ldots,k_{p_{n+1}})q^{-2(l-1)} \\
&\quad\times\exp\left(\mathrm{i}\sum_{j=1}^{l-1} k_{p_j}x_j+\mathrm{i}\sum_{j=l+1}^{n+1} k_{p_j}x_{j-1}\right)\frac{[q\exp(\mathrm{i}k_{p_l})]^{x_{l-1}+1}-[q\exp(\mathrm{i}k_{p_l})]^{x_l}}{1-q\exp(\mathrm{i}k_{p_l})} \\
&\quad+\sum_p \varepsilon_p A(k_{p_1},\ldots,k_{p_{n+1}})\exp\left(\mathrm{i}\sum_{j=2}^{n+1} k_{p_j}x_{j-1}\right)\frac{[q\exp(\mathrm{i}k_{p_1})]-[q\exp(\mathrm{i}k_{p_1})]^{x_1}}{1-q\exp(\mathrm{i}k_{p_1})}.
\end{aligned}
\tag{3.6}
$$

Since

$$
\begin{aligned}
&A(k_{p_1},\ldots,k_{p_{n+1}})\alpha(k_{p_1})-A(-k_{p_1},\ldots,k_{p_{n+1}})\alpha(-k_{p_1})=0 \\
&A(k_{p_1},\ldots,k_{p_{n+1}})\beta(-k_{p_{n+1}})-A(k_{p_1},\ldots,-k_{p_{n+1}})\beta(k_{p_{n+1}})=0
\end{aligned}
$$

it is easy to show that

$$
\begin{aligned}
&\sum_p \varepsilon_p A(k_{p_1},\ldots,k_{p_{n+1}})\exp\left(\mathrm{i}\sum_{j=1}^{n} k_{p_j}x_j\right)\frac{[q\exp(\mathrm{i}k_{p_{n+1}})]^{N+1}}{1-q\exp(\mathrm{i}k_{p_{n+1}})}=0 \\
&\sum_p \varepsilon_p A(k_{p_1},\ldots,k_{p_{n+1}})\exp\left(\mathrm{i}\sum_{j=2}^{n+1} k_{p_j}x_{j-1}\right)\frac{[q\exp(\mathrm{i}k_{p_1})]}{1-q\exp(\mathrm{i}k_{p_1})}=0.
\end{aligned}
$$

We can rewrite (3.6) as

$$q^{(N-2n+1/2)}g(x_1,\ldots,x_n)$$
$$=\sum_{l=1}^{n-1}\sum_p \varepsilon_p A(k_{p_1},\ldots,k_{p_{n+1}})\exp\left(\mathrm{i}\sum_{j=1}^{l-1}k_{p_j}x_j+\mathrm{i}\sum_{j=l+2}^{n+1}k_{p_j}x_{j-1}\right)$$
$$\times\frac{q^{-2l+x_l+1}\exp[\mathrm{i}(k_{p_l}+k_{p_{l+1}})x_l]}{[1-q\exp(\mathrm{i}k_{p_{l+1}})][1-q\exp(\mathrm{i}k_{p_l})]}$$
$$\times\{(q^2+1)\exp(\mathrm{i}k_{p_{l+1}})-q-q\exp[\mathrm{i}(k_{p_{l+1}}+k_{p_l}]\}. \tag{3.7}$$

Summing over the permutation of k_{p_l} and $k_{p_{l+1}}$, and noticing that

$$\frac{A(k_{p_1},\ldots,k_{p_l},k_{p_{l+1}},\ldots,k_{p_{n+1}})}{A(k_{p_1},\ldots,k_{p_{l+1}},k_{p_l},\ldots,k_{p_{n+1}})}=\frac{(q^2+1)\exp(\mathrm{i}k_{p_l})-q-q\exp[\mathrm{i}(k_{p_l}+k_{p_{l+1}})]}{(q^2+1)\exp(\mathrm{i}k_{p_{l+1}})-q-q\exp[\mathrm{i}(k_{p_l}+k_{p_{l+1}})]} \tag{3.8}$$

it is easy to check that the summation vanishes:

$$g(x_1,\ldots,x_n)=0\Rightarrow S_+|n+1\rangle_{\mathrm{BA}}=0. \tag{3.9}$$

Thus $|n\rangle_{\mathrm{BA}}$ is a HWS of $SU_q(2)$ with $j=(N-2n)/2$. Applying S_- on $|n\rangle_{\mathrm{BA}}$ successively, we obtain $N-2n+1$ states having the same energy. They constitute an $(N-2n+1)$-dimensional irreducible representation of $SU_q(2)$ for a generic q. For such an N spin-$\frac{1}{2}$ system, by counting the number of states, one can show that there exist $\Gamma_n^N=C_n^N-C_{n-1}^N$ independent HWSs of $SU_q(2)$ with $j=(N-2n)/2$ for a given $n\leqslant N/2$. Pasquier and Saleur indicated that there also exist Γ_n^N linearly independent BA states with n downward spins. It is easy to check that

$$\sum_{n=0}^{\{N/2\}}(N-2n+1)\Gamma_n^N=2^N \tag{3.10}$$

where $\{N/2\}$ is the integral part of $N/2$. Thus the states in all irreducible representations having BA states as HWSs are complete. (The spin-$\frac{1}{2}$ particle has two independent spin states, and thus the N spin-$\frac{1}{2}$ system has 2^N spin states.) Consequently the open *XXZ* spin chain eigenstate space is classified to the irreducible representations of $SU_q(2)$, with different eigenenergies corresponding to different irreducible representations. It is important to point out that so far we have excluded the case for q being a root of unity. For the case of q being a root of unity, which we denote as q_0 for clarity, things are far more complicated.

4. Solutions of the BA equation and null states for q_0

In a BA state, when q is not a root of unity, the impulsions k are all distinct, or will have zero amplitude. On the other hand, the function has no reflected wave, if $\mathrm{e}^{\mathrm{i}k}=q_0$. However, when $q\to q_0$, we may have BA states with some impulsions $k\to\gamma_0$, where $q_0=\mathrm{e}^{\mathrm{i}\gamma_0}$. We study the case in which there can be m identical impulsions $k\to\gamma_0$.

Write the BA equations as

$$\mathrm{e}^{\mathrm{i}2Nk_j}=\prod_{\substack{l=1\\l\neq j}}^{n}\frac{(\mathrm{e}^{-\mathrm{i}k_j}+\mathrm{e}^{\mathrm{i}k_l}-\Delta)(\mathrm{e}^{-\mathrm{i}k_j}+\mathrm{e}^{-\mathrm{i}k_l}-\Delta)}{(\mathrm{e}^{\mathrm{i}k_j}+\mathrm{e}^{\mathrm{i}k_l}-\Delta)(\mathrm{e}^{\mathrm{i}k_j}+\mathrm{e}^{-\mathrm{i}k_l}-\Delta)}\mathrm{e}^{\mathrm{i}2k_j}\qquad(j=1,\ldots,n) \tag{4.1}$$

where $\Delta\equiv q+q^{-1}$.

Suppose when $q \to q_0$, with $q_0^p = -1$, there exists a solution $\{k_j\}$ to equations (4.1) with n distinct ks. We study if $n+m$ transcendent equations may accommodate a solution with m ks $\to \gamma_0$, which are denoted as γ_j, and the rest of the impulsions are $\{k_j'\}$, $k_j' \to k_j$.

We have from (4.1)

$$e^{i2(N-n-m+1)k_j'} = \prod_{\substack{l=1\\ l\neq j}}^{n} \frac{(e^{-ik_j'}+e^{ik_l'}-\Delta)(e^{-ik_j'}+e^{-ik_l'}-\Delta)}{(e^{ik_j'}+e^{ik_l'}-\Delta)(e^{ik_j'}+e^{-ik_l'}-\Delta)} \times \prod_{l=1}^{m} \frac{(e^{-ik_j'}+e^{i\gamma_l}-\Delta)(e^{-ik_j'}+e^{-i\gamma_l}-\Delta)}{(e^{ik_j'}+e^{i\gamma_l}-\Delta)(e^{ik_j'}+e^{-i\gamma_l}+\Delta)} \tag{4.2}$$

$$e^{i2(N-n-m+1)\gamma_j} = \prod_{\substack{l=1\\ l\neq j}}^{m} \frac{(e^{-i\gamma_j}+e^{-i\gamma_l}-\Delta)(e^{-i\gamma_j}+e^{i\gamma_l}-\Delta)}{(e^{+i\gamma_j}+e^{-i\gamma_l}-\Delta)(e^{i\gamma_j}+e^{i\gamma_l}-\Delta)} \times \prod_{l=1}^{n} \frac{(e^{-i\gamma_j}+e^{ik_l'}-\Delta)(e^{-i\gamma_j}+e^{-ik_l'}-\Delta)}{(e^{i\gamma_j}+e^{ik_l'}-\Delta)(e^{i\gamma_j}+e^{-ik_l'}-\Delta)}. \tag{4.3}$$

When $\gamma \to \gamma_0$, (4.2) becomes

$$e^{i2(N-n-m+1)k_j} = \prod_{\substack{l=1\\ l\neq j}}^{n} \frac{(e^{-ik_j}+e^{-ik_l}-\Delta)(e^{-ik_j}+e^{ik_l}-\Delta)}{(e^{ik_j}+e^{-ik_l}-\Delta)(e^{ik_j}+e^{ik_l}-\Delta)} \prod_{l=1}^{m} e^{-2ik_j} \tag{4.4}$$

and (4.4) is equivalent to (4.1). Let $\gamma_j = \gamma + \varepsilon_j$ and $\gamma - \gamma_0 = \lambda$; when $\lambda \to 0$, $\varepsilon \to 0$, from (4.3), we have

$$(-1)^{m-1} e^{i2(N+1-m-2n)\gamma_j}(1+O(\varepsilon_j)) = \prod_{\substack{l=1\\ l\neq j}}^{m} \frac{(e^{-i\gamma_j}+e^{i\gamma_l}-\Delta)}{(e^{i\gamma_j}+e^{-i\gamma_l}-\Delta)}. \tag{4.5}$$

The zero-order equations imply that when

$$q_0^{2(N-m-2n+1)} = 1 \tag{4.6a}$$

we have

$$\lim_{\varepsilon\to 0} \prod_{\substack{l=1\\ l\neq j}}^{m} \frac{\varepsilon_j - q_0^2\varepsilon_l}{\varepsilon_l - q_0^2\varepsilon_j} = (-1)^{m-1}. \tag{4.6b}$$

One obvious solution of (4.6) is

$$\varepsilon_j = \varepsilon\, e^{i2\pi j/m}. \tag{4.7}$$

Now we construct a BA state $|n, m\rangle_{BA}$ by using the solutions of (4.6) and the following relations:

$$\beta(-\gamma_j) = (1-q^2)q^{-(N+1)} + O(\varepsilon) \tag{4.8a}$$

$$\frac{B(-k_j, \gamma_l)\, e^{-i\gamma_l}}{B(-\gamma_l, k_j)\, e^{-ik_j}} = -q^{-2} + O(\varepsilon). \tag{4.8b}$$

According to the definition of wavefunction f, and taking into account (4.8) we may have the following relation after a lengthy calculation (see the appendix):

$$\sum_{\{\nu_l\}} C' q^{-2\Sigma\nu_l} f(x_{i_1},\ldots,x_{i_n})\exp[i\gamma(x_{j_1}+\ldots+x_{j_m})]\left(\prod_{i<j}(\varepsilon_i-\varepsilon_j)\right)[m]!(1+O(\varepsilon))$$
$$=f(x_1,x_2,\ldots,x_{n+m}) \tag{4.9}$$

where ν_l is the number of ks whose corresponding x_i are smaller than the x_{j_l} ($l=1,\ldots,m$) corresponding to γ_{p_l}, and $f(x_{i_1},\ldots,x_{i_n})$ are the coefficients of $|n\rangle_{\text{BA}}$ while $\{x_{i_1}<x_{i_2}\ldots<x_{i_n}\}$ is a subset of $\{x_1<x_2\ldots<x_{n+m}\}$. Consequently we can construct the state $|n,m\rangle_{\text{BA}}$ as long as

$$\varepsilon_i\neq\varepsilon_j \qquad (i\neq j) \qquad [m]!\neq 0 \qquad C'\neq 0.$$

When $q=q_0$, (4.1) and (4.4) have the solution $\{k_j\}$ with distinct ks. Applying S_- we obtain a linear space. It is easy to show that the state $(S_-)^m|n\rangle_{\text{BA}}$ is a null state if

$$q_0^{2(N-2n-m+1)}=1 \tag{4.10}$$

or $N-2n-m+1=0$ (mod P), $q_0^P=-1$.

From (2.3), we may construct $(S_-)^m|n\rangle_{\text{BA}}$ by inserting m x_{j_l}s into the row $x_1<\ldots<x_n$, and each insertion produces a factor,

$$q^{(1/2)(x-1-2\theta_l)-(1/2)(N-x-2(n+t-\theta_l))}=q^{x-2\theta_l}q^{t-(1/2)(N-2n+1)}$$

where θ_l is the number of xs before x_{j_l}, and t is the number of early inserted x'_{j_i}. Thus we have

$$(S_-)^m|x_1,\ldots,x_n\rangle$$
$$=\sum_{\{x_l\}}|x_1,\ldots,x_n;x_{j_1},\ldots,x_{j_m}\rangle\{q^{-(1/2)(N-2n-m+2)m}\}q^{x_{i_1}+x_{i_2}+\ldots+x_{i_m}}$$
$$\times\left(\sum_p q^{-2\theta_l}\right). \tag{4.11}$$

It is easy to show for permutatively inserting the $\{x_{j_l}\}$, with fixed $x_1,\ldots,x_n$,

$$\sum_p q^{-2\theta_l}=q^{-2\Sigma\nu_l}\times 1\times(1+q^{-2})(1+q^{-2}+q^{-4})\ldots(1+q^{-2}+\ldots+q^{-2(m-1)})$$

and we have

$$(S_-)^m|x_1,\ldots,x_n\rangle$$
$$=\sum_{\{x_j\}}|x_1,\ldots,x_n;x_{j_1},\ldots,x_{j_m}\rangle q^{-(m/2)(N-2n-m+2)}$$
$$\times q^{x_{j_1}+\ldots+x_{i_m}}q^{-2\Sigma\nu_l}[m]!q^{(m/2)(m-1)} \tag{4.12}$$

giving

$$(S_-)^m|n\rangle_{\text{BA}}=\sum_{\{x\}}\sum_{\{\nu_l\}}f(x_{i_1},\ldots,x_{i_n})\exp[i\gamma(x_{j_1}+\ldots+x_{j_m})]$$
$$\times q^{-2\Sigma\nu_l}[m]!q^{-(m/2)(N-2n+1)}|x_1,\ldots,x_{n+m}\rangle. \tag{4.13}$$

Comparing (4.9) and (4.13), when $N-2n-m+1=0$ (mod P), and $[m]!\neq 0$, $C'\neq 0$, we have a $|n,m\rangle_{\text{BA}}\equiv\{C'\Pi_{i<j}(\varepsilon_i-\varepsilon_j)\}^{-1}|n,m\rangle_{\text{BA}}$,

$$|n,m\rangle_{\text{BA}'}\to c^{-1}(q_0)(S_-)^m|n\rangle_{\text{BA}} \tag{4.14}$$

where $\{k_j\}$ in $|n\rangle_{\text{BA}}$ is just the set of distinct ks different from γ_0 in $|n,m\rangle_{\text{BA}}$.

From (4.14), we can see that it is the HWS of the irreducible representation, i.e. $|n, m\rangle_{BA'}$ coincides with $(S_-)^m|n\rangle_{BA}$, a null state of another representation, when $q \to q_0$. The space constructed from $|n, m\rangle_{BA'}$ by S_- is in fact a subspace of that constructed from $|n\rangle_{BA}$.

Because of the overlapping of the states, there appears the problem of 'missing states'. Consequently, the $SU_q(2)$ representation is changed, and an indecomposable representation emerges. To construct this type of representation, we need to make up for the 'missing states'.

5. Type I representation of $SU_q(2)$

In the last section we have studied the solution of the BA equation in detail, and showed that the solution with n distinct k_js and m γ_0s can be seen as the solution of the BA equation in the limit $q \to q_0$. The corresponding states $|n, m\rangle_{BA'}$ and $c^{-1}(S_-)^m|n\rangle_{BA}$ are identical as $q \to q_0$. Moreover, they are null states.

Because of this, the space generated from BA states by S_- is no longer complete. To make up for the compensation, we look for a state satisfying the following relations:

$$S_+|b\rangle = S_-^{m-1}|n\rangle_{BA} C_1 \tag{5.1}$$

$$H|b\rangle = E_n|b\rangle + C_2 S_-^m|n\rangle_{BA} \tag{5.2}$$

where C_1, C_2 are constants and $E_n(q_0) = E_{n+m}(q_0)$ is the energy expectation value at $q = q_0$ such that under S_-, $|b\rangle$ generates the 'missing states'. In this case, (5.2) implies that the Hamiltonian is not completely diagonalizable.

We require that $|b\rangle$ is orthogonal to the states of other representations. Thus we start directly from $|n\rangle_{BA}$ when q is not a root of unity. On the other hand, $(S_-)^m|n\rangle_{BA}$ and $|n, m\rangle_{BA}$ are proportional to each other when $q \to q_0$. Thus we define

$$|b\rangle = \lim_{q\to q_0} \frac{S_-^m|n\rangle_{BA} - c(q_0)|n, m\rangle_{BA'}}{q - q_0} = \frac{d}{dq}(S_-^m|n\rangle_{BA} - c(q_0)|n, m\rangle_{BA'})_{q_0}. \tag{5.3}$$

Subsequently we show that $|b\rangle$ defined above satisfies (5.1) and (5.2). From (2.5), we have

$$S_+(S_-)^m|n\rangle_{BA} = [m][N - 2n - m + 1]S_-^{m-1}|n\rangle_{BA}. \tag{5.4}$$

Taking the derivative at $q = q_0$ and noticing that $[N - 2n - m + 1]$ is $\sim O(q - q_0)$, we get

$$\left(\frac{d}{dq} S_+\right)_{q_0} (S_-^m|n\rangle_{BA})_{q_0} + (S_+)_{q_0}\left(\frac{d}{dq} S_-^m|n\rangle_{BA}\right)_{q_0}$$
$$= [m]_{q_0}(N - 2n - m + 1)\frac{q_0^{N-2n-m} + q_0^{-N+2n+m-2}}{q_0 - q_0^{-1}}(S_-^{m-1}|n\rangle_{BA})_{q_0} \tag{5.5}$$

where $(d/dq\, S_+)_{q_0}$ is the derivative of S_+ at $q = q_0$, obtained by differentiating (2.3) with respect to q.

Since

$$\frac{d}{dq}(S_+|n, m\rangle_{BA'})_{q_0} = \left(\frac{d}{dq} S_+\right)_{q_0} |n, m\rangle_{BA'} + (S_+)_{q_0}\left(\frac{d}{dq}|n, m\rangle_{BA'}\right)_{q_0} = 0$$

we have

$$(S_+)_{q_0}\left(\frac{\mathrm{d}}{\mathrm{d}q}S_-^m|n\rangle_{\mathrm{BA}}-c(q_0)\frac{\mathrm{d}}{\mathrm{d}q}|n,m\rangle_{\mathrm{BA}'}\right)_{q_0}$$

$$=(S_+)_{q_0}|b\rangle$$

$$=[m]\frac{q_0^{N-2n+m}+q_0^{-N+2n+m+2}}{q_0-q_0^{-1}}(N-2n-m+1)(S_-^{m-1}|n\rangle_{\mathrm{BA}})_{q_0} \tag{5.6}$$

i.e. $|b\rangle$ thus defined satisfies (5.1). In a similar way we can show that $|b\rangle$ satisfies (5.2), and give the results

$$H|n\rangle_{\mathrm{BA}}=E_n|n\rangle_{\mathrm{BA}} \qquad H|n,m\rangle_{\mathrm{BA}}=E_{n+m}|n,m\rangle_{\mathrm{BA}'}$$
$$H|b\rangle=E_n(q_0)|b\rangle+\frac{\mathrm{d}}{\mathrm{d}q}(E_n-E_{n+m})(S_-)^m|n\rangle_{\mathrm{BA}}. \tag{5.7}$$

Thus the newly defined $|b\rangle$ can make up the 'missing states'. A quasidiagonal H keeps diagonal elements of H unchanged. And the off-diagonal elements are non-vanishing. This representation is indecomposable.

From the above procedure we can see that there always exits a state $|b\rangle$, such that applying S_- on $|n\rangle_{\mathrm{BA}}$ and $|b\rangle$, we can have the corresponding state space of the representation. As $(q_0)^p=-1$, it is easy to show $(S_-)^p=0$. We use $(S_-)^p/[p]$ to construct the state. Explicitly, we define (notice $j-j'=m$ and $j+j'+1=0 \bmod p$)

$$|j-lp-r,A\rangle=(S_-)^{lp+r}/[p]^l|n\rangle$$
$$|j'-lp-r,B\rangle=(S_-)^{lp+r}/[p]^l|b\rangle$$

for $0\leqslant r<p$. We then have

$$S_z|M,A\rangle=M|M,A\rangle \qquad |M-1,A\rangle=\frac{S_-}{[p]^\alpha}|M,A\rangle$$

$$S_z|M,B\rangle=M|M,B\rangle \qquad |M-1,B\rangle=\frac{S_-}{[p]^\beta}|M,B\rangle$$

where

$$\alpha=\begin{cases}1 & \text{when } j-M+1=0(\operatorname{mod} p)\\ 0 & \text{otherwise}\end{cases}$$

$$\beta=\begin{cases}1 & \text{when } j'-M+1=0(\operatorname{mod} p)=-(j+M(\operatorname{mod} p))\\ 0 & \text{otherwise.}\end{cases}$$

When $r+1<p$, the action of S_+ gives

$$S_+|j'-lp-r-1,B\rangle$$
$$=S_+(S_-)^{lp+r+1}/[p]^l|b\rangle$$
$$=(S_-)^{lp+r+1}/[p]^lS_+|b\rangle+[S_+,(S_-)^{lp+r+1}]/[p]^l|b\rangle$$
$$=(S_-)^{lp+r+1}/[p]^lc_1(S_-)^{m-1}|n\rangle_{\mathrm{BA}}+\frac{[lp+r+1]}{[p]^l}(S_-)^{lp+r}[2S_z-lp-r]|b\rangle$$
$$=(S_-)^{lp+r+m}/[p]^lc_1|n\rangle_{\mathrm{BA}}+[lp+r+1][2j'-lp-r](S_-)^{lp+r}/[p]^l|b\rangle$$
$$=Fc_1|j-lp-r-m,A\rangle+[lp+r+1][2j'-lp-r]|j'-lp-r,B\rangle.$$

where $F=[p]$ for $r+m\geqslant p$ and $F=1$ for $r+m<p$. When $r+1=p$, we have

$$
\begin{aligned}
S_+|j'-lp-r-1, B\rangle
&= S_+(S_-)^{(l+1)p}/[p]^{l+1}|b\rangle \\
&= (S_-)^{(l+1)p+m-1}/[p]^{l+1}c_1|n\rangle_{\mathrm{BA}}+\frac{[(l+1)p]}{[p]}[2j'-lp-r]|j'-lp-r, B\rangle \\
&= c_1|j-(l+1)p-m+1, A\rangle+\frac{[(l+1)p]}{[p]}[2j'-lp-r]|j'-lp-r, B\rangle.
\end{aligned}
$$

Other relations of $S_\pm$ on $|B\rangle$ and $|A\rangle$ can be similarly obtained. Thus $\{|B\rangle, |A\rangle\}$ constitute a representation of $\mathrm{SU}_q(2)$.

Notice that

$$
\begin{aligned}
\langle j', B|j-m, A\rangle
&= \left(\frac{1}{\Delta q}\right)\{|S_-^m|n\rangle_{\mathrm{BA}}-c|n+m\rangle_{\mathrm{BA}}\}^+S_-^m|n\rangle_{\mathrm{BA}} \\
&= \left(\frac{1}{\Delta q}\right)_{\mathrm{BA}}\langle n|S_+^m S_-^m|n\rangle_{\mathrm{BA}} \\
&= [1][2]\dots[m-1][2j][2j-1]\dots[2j-m+2]\frac{[2j-m+1]}{\Delta q}\langle n|n\rangle \\
&= \mathrm{const}\neq 0
\end{aligned}
$$

and

$$
\begin{aligned}
\langle M-1, B|M-1, A\rangle
&= \langle M, B|S_+S_-|M, A\rangle/\{[p]^{\alpha+\beta}\} \\
&= \left\langle M, B\left|\frac{[j+M][j-M+1]}{[p]^{\alpha+\beta}}\right|M, A\right\rangle \\
&= \frac{[j+M]}{[p]^\beta}\frac{[j-M+1]}{[p]^\alpha}\langle M, B|M, A\rangle.
\end{aligned}
$$

Since

$$\frac{[j+M]}{[p]^\beta}\frac{[j-M+1]}{[p]^\alpha}\neq 0 \qquad \text{for } j\geqslant M>-j$$

we have $\langle M, B|M, A\rangle\neq 0$, giving $|M, B\rangle, |M, A\rangle\neq 0$ for $j'\geqslant M\geqslant -j$ (when $M<-j', |M, B\rangle\sim|M, A\rangle$).

So far we have found an approach to make up 'missing states' and to generate a representation of $\mathrm{SU}_q(2)$. The representation is not irreducible but indecomposable, i.e. the type I representation of $\mathrm{SU}_q(2)$. For an N spin lattice system, there exist variant sets of the solutions $\{k_j\}$ of the BA equation. Some of them have only distinct impulsions k_js, while others have distinct impulsions k_j and identical impulsions γ_0, and can be classified into two classes. In the first case we cannot find another solution which has some ks and m identical γ_0s for a state in this class. For the solution in the second class, we can find a pair of solutions which generate an indecomposable representation. Therefore, we have given an approach to construct type I and II representations of $\mathrm{SU}_q(2)$.

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Appendix

For a base vector $|x_1, \ldots, x_{n+m}\rangle$ we derive $f(x_1, \ldots, x_{n+m})$. To perform the permutations and negations of n ks and m γs to match $n+m$ xs, we first choose m xs $x_{j_1} < x_{j_2} < \ldots < x_{j_m}$ for m γs. At the same time we have n xs $x_{i_1} < x_{i_2} < \ldots < x_{i_n}$ for n ks. This is equivalent to giving a set of numbers $0 \leqslant \nu_1 \leqslant \nu_2 \leqslant \ldots \leqslant \nu_m \leqslant n$, where ν_l denotes the number of x_{i_s}s smaller than x_{j_l}. Next we perform the permutations and negations of ks to match the n x_{i_s}s, and the permutations and negations of γs to match the m x_{j_l}s. We have k_{p_s} corresponding to x_{i_s} and γ_{p_l} corresponding to x_{j_l}, and have

$$\exp\left(\mathrm{i} \sum k_{p_s} x_{i_s} + \mathrm{i} \sum \gamma_{p_l} x_{j_l}\right) \equiv \mathrm{e}^{\mathrm{i}kx} \tag{A1a}$$

$$\sum_p = \sum_{\{\nu_l\}} \sum_{\text{per. of } k} \sum_{\text{neg. of } k} \sum_{\text{per. of } \gamma} \sum_{\text{neg. of } \gamma} \equiv \Sigma_1 \Sigma_2 \Sigma_3 \Sigma_4 \Sigma_5 \tag{A1b}$$

$$\begin{aligned} \varepsilon_p &= (\varepsilon_p)_{\{\nu_l\}} (\varepsilon_p)_{\text{per. of } k} (\varepsilon_p)_{\text{neg. of } k} (\varepsilon_p)_{\text{per. of } \gamma} (\varepsilon_p)_{\text{neg. of } \gamma} \\ &\equiv (\varepsilon_p)_1 (\varepsilon_p)_2 (\varepsilon_p)_3 (\varepsilon_p)_4 (\varepsilon_p)_5. \end{aligned} \tag{A1c}$$

Since $\beta(\gamma_j) = \mathrm{O}(\varepsilon)$, in the last step we therefore need only to consider the permutations of γs. Also, in the factor $\mathrm{e}^{\mathrm{i}kx}$, we can use $\exp(\mathrm{i}\gamma x_{j_l})$ for $\exp(\mathrm{i}\gamma_{p_l} x_{j_l})$ with a relative error $\sim \mathrm{O}(\varepsilon)$.

We have from (2.10), (2.11)

$$\begin{aligned} f(x_1, \ldots, x_{n+m}) & \\ &\equiv f(x_{i_1}, \ldots, x_{i_n}; x_{j_1}, \ldots, x_{j_m}) \\ &= \Sigma_1 \Sigma_2 \Sigma_3 \Sigma_4 (\varepsilon_p)_1 (\varepsilon_p)_2 (\varepsilon_p)_3 (\varepsilon_p)_4 \prod_{x_{i_s} < x_{j_l}} B(-k_{p_s}, \gamma_{p_l}) \exp(-\mathrm{i}\gamma_{p_l}) \\ &\quad \times \prod_{x_{i_s} > x_{j_l}} B(-\gamma_{p_l}, k_{p_s}) \exp(-\mathrm{i}k_{p_s}) \prod_s \beta(-k_{p_s}) \\ &\quad \times \prod_{x_{i_s} < x_{i_{s'}}} B(-k_{p_s}, k_{p_{s'}}) \exp(-\mathrm{i}k_{p_{s'}}) \prod_l \beta(-\gamma_{p_l}) \\ &\quad \times \prod_{x_{j_l} < x_{j_{l'}}} B(-\gamma_{p_l}, \gamma_{p_{l'}}) \exp(-\mathrm{i}\gamma_{p_{l'}}) \, \mathrm{e}^{\mathrm{i}kx} (1 + \mathrm{O}(\varepsilon)). \end{aligned} \tag{A2}$$

Noticing that

$$B(-k_s, \gamma_l)\, \mathrm{e}^{-\mathrm{i}\gamma_l} = B(k_s, \gamma_l)\, \mathrm{e}^{-\mathrm{i}\gamma_l}(1 + \mathrm{O}(\varepsilon))$$

$$B(-\gamma_l, k_s)\, \mathrm{e}^{-\mathrm{i}k_s} = B(-\gamma_l, -k_s)\, \mathrm{e}^{\mathrm{i}k_s}(1 + \mathrm{O}(\varepsilon))$$

$$\begin{aligned} &\prod_{x_{i_s} < x_{j_l}} B(-k_{p_s}, \gamma_{p_l}) \exp(-\mathrm{i}\gamma_{p_l}) \prod_{x_{i_s} > x_{j_l}} B(-\gamma_{p_l}, k_{p_s}) \exp(-\mathrm{i}k_{p_s}) \\ &\quad = \prod_{l,s} B(-\gamma_{p_l}, k_{p_s}) \exp(-\mathrm{i}kp_s)(-q^{-2})^{\nu_1 + \ldots + \nu_m} \end{aligned} \tag{A3}$$

we can move $\Sigma_2 \Sigma_3 (\varepsilon_p)_2 (\varepsilon_p)_3$ over

$$\prod B(-k_{p_s}, \gamma_{p_l}) \exp(-\mathrm{i}\gamma_{p_l}) \prod B(-\gamma_{p_l}, k_{p_s}) \exp(-\mathrm{i}k_{p_s})$$

and obtain

$$
\begin{aligned}
f=\Sigma_1(\varepsilon_p)_1(-q^{-2})^{\Sigma\nu_l} \\
&=\prod_{l,s} B(-\gamma_l, k_s)\,\mathrm{e}^{-\mathrm{i}k_s} \sum_{\{\nu_l\}} (q^{-2})^{\Sigma\nu_l}\Sigma_2\Sigma_3(\varepsilon_p)_2(\varepsilon_p)_3 \prod_s \beta(-k_{p_s}) \\
&\quad\times \prod_{x_{i_s}<x_{i_{s'}}} B(-k_{p_s}, k_{p_{s'}}) \exp(-\mathrm{i}k_{p_{s'}}) \exp[\mathrm{i}(k_{p_1}x_{i_1}+\ldots+k_{p_n}x_{i_n})] \\
&\quad\times \Sigma_4(\varepsilon_p)_4 \prod_l \beta(-\gamma_l) \prod_{x_{i_l}<x_{i_{l'}}} B(-\gamma_{p_l}, \gamma_{p_{l'}}) \\
&\quad\times \exp(-\mathrm{i}\gamma_{p_{l'}}) \exp[\mathrm{i}\gamma(x_{j_1}+\ldots+x_{j_m})]\{1+\mathrm{O}(\varepsilon)\} \\
&= C'' \sum_{\{\nu_l\}} (q^{-2})^{\Sigma\nu_l} f(x_{i_1},\ldots,x_{i_n}) \\
&\quad\times \left[\Sigma_4(\varepsilon_p)_4 \prod_{x_{i_l}<x_{i_{l'}}} (\varepsilon_{p_l}-q^2\varepsilon_{p_{l'}})\right] \\
&\quad\times \exp[\mathrm{i}\gamma(x_{j_1}+\ldots+x_{j_m})](1+\mathrm{O}(\varepsilon))
\end{aligned}
\tag{A4}
$$

where

$$
C''=\prod_{ls} B(-\gamma_l, k_s)\,\mathrm{e}^{-\mathrm{i}k_s} \prod_l \beta(-\gamma_l)[(-\mathrm{i})(1-q^2)]^m.
$$

We can show that

$$
\Sigma_4(\varepsilon_p)_4 \prod (\varepsilon_{p_l}-q^2\varepsilon_{p'_l}) = \prod_{i<j} (\varepsilon_i-\varepsilon_j)[m]!\,q^{(m/2)(m-1)}
\tag{A5}
$$

and finally obtain

$$
f=\sum_{\{\nu_l\}} C'(q^{-2})^{\Sigma\nu_l} f(x_{i_1},\ldots,x_{i_n}) \prod_{i<j} (\varepsilon_i-\varepsilon_j)[m]!\exp\left(\mathrm{i}\gamma\sum_l x_{j_l}\right).
\tag{A6}
$$

References

[1] Pasquier N and Saleur H 1990 *Nucl. Phys.* B **330** 523

[2] Gaudin M 1983 *La Fonction d'Onde de Bethe* (Paris: Masson)

[3] Takahtazhan L A and Faddeev L D 1979 *Uspeki Math: Nauk.* **34** 5, 13

[4] Lowenstein J H 1982 Introduction to the Bethe-Ansatz approach in 1+1-dimensional models *Recent Advances in Field Theory and Statistical Mechanics* ed J B Zuber and R Stora (Amsterdam: North-Holland)

[5] Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (New York: Academic)

[6] Takahtazhan L A and Faddeev L D 1981 *Zap. Nauch. Semin. LOMI* **109** 134

[7] Yang C N and Yang C P 1966 *Phys. Rev.* **150** 321, 327

[8] Izergin A G and Korepin V E 1982 *Lett. Math. Phys.* **6** 283

[9] Alcarez F C, Barber M N, Betchelor M T, Baxter R J and Quispel G R W 1987 *J. Phys. A: Math. Gen.* **20** 6397

[10] Sklyanin E K 1988 *J. Phys. A: Math. Gen.* **21** 2375

[11] Bo-Yu Hou, Bo-Yuan Hou and Zhong-Qi Ma 1990 *Preprint* IC/90/386

[12] Karowski M 1989 *Preprint* ETH-TH-89-43

[13] Mezincescu L and Nepomechie R I 1989 *Preprint* UMTG-158